A Gentle Introduction to the Calculus of Sustainable Income: What Is Your Retirement RisQuotient?

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Abstract: *A little over a year ago, on January 1, 2006, the first American baby boomer turned 60. These birthdays are expected to continue at the rate of one per 7-10 seconds over the next 20 years. In anticipation of this demographic wave the financial services industry is bracing for the retirement income revolution, and one of the critical issues is how to build a portfolio that will provide a sustainable income flow over the uncertain length and cost of the human lifecycle. Indeed, a number of recent articles have gained notoriety by advocating spending rates in the 4-6% vicinity as being sustainable for portfolios that contain 70-90% equity exposure.*

But prudent risk management involves more than just controlled consumption, and this article deliberately avoids advocating a particular spending rate. Instead it provides an overview of the analytic relationship or calculus among the three key risk variables that determine income sustainability. These are brought together by linking investment characteristics, spending rates, and longevity risk, to provide what is coined the Retirement RisQuotient.[1] And, while statistical formulas will never capture the complex nuances of retirement reality, there are a variety of intuitive insights that can be gleaned from this summary number. Moreover, this calculus illustrates how products with longevity insurance (e.g., life annuities) and downside protection (e.g., embedded put options) can increase the sustainability by reducing the Retirement RisQuotient.

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Introduction and Motivation: Boomer Wants Income

The last few years have seen intense research around the topic of retirement income planning. One of the most widely studied issues is how to build a portfolio that will provide a sustainable income flow over the uncertain length and cost of the human lifecycle. Not surprisingly, a number of recent articles have argued that one of the key components to a sustainable retirement is to simply avoid spending too much. A variety of authors have advocated spending rates in the 4-6% vicinity as being sustainable for portfolios that contain 70% to 90% equity exposure. To be clear, a spending rate of x percent is meant to imply withdrawing $x per original $100 nest egg on an annual basis, with each year's spending being adjusted for consumer price inflation.[2]

For example, Bengen; Ho, Milevsky, and Robinson; Cooley, Hubbard, and Walz; Pye; Milevsky; Ameriks, Veres, and Warshawsky; and Guyton have all created financial experiments incorporating historical, simulated, and scrambled returns to quantify the sustainability of various *ad hoc* spending policies and consumption rates for retired individuals.[3] As a number of financial commentators have emphasized, the problem with such studies is that (1) these simulations are often difficult to replicate by financial practitioners, (2) many conduct only a minimal number of simulations, and (3) most provide little pedagogical intuition on the financial tradeoff between retirement risk and return. More importantly, they focus too much attention on the relationship between spending habits and the composition of the investment portfolio, yet they neglect the many other factors that determine retirement income sustainability. For example, important questions such as: "What are the roles of variable payout (or income) annuities and their guaranteed riders in a sustainable portfolio?" are not addressed in any of these studies.

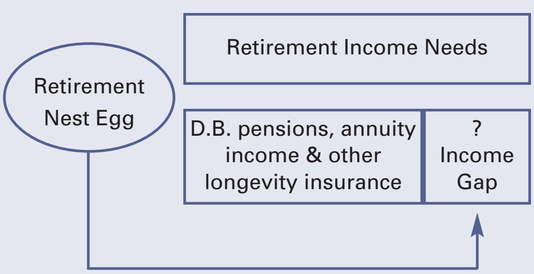
Therefore, to differentiate this article from the recent simulation-based debate around prudent retirement spending, this article will deliberately avoid advocating a preferred portfolio withdrawal rate. Instead, it provides an overview of the analytical relationship between the key risk variables that determine sustainability. These ingredients are brought together by linking portfolio parameters, spending rates, and longevity risk to an analytic probability of retirement ruin. More specifically, a number of key variables are identified that will determine a retiree's Retirement RisQuotient.

While statistical formulas will never capture the complex nuances of retirement reality, there are a variety of intuitive insights that can be gleaned from this summary Retirement RisQuotient metric. Moreover, this calculus helps illustrate how products with longevity insurance (e.g., life annuities) and downside protection (e.g., the put options embedded within variable annuities) can increase the sustainability of retirement.

The remainder of this article is organized as follows. The basic formula that helps us estimate ruin and sustainability is presented first. Although the underlying mathematics are quite complicated and are described elsewhere, the main formula itself can be implemented using a function that is easily available in Microsoft Excel. Next, a short-cut method for arriving at the Retirement RisQuotient using two summary variables is provided. Some caveats and warnings about the underlying assumptions are discussed, and then illustrations show how this risk measurement system can be extended to include life annuities and various downside protected portfolio strategies, many of which are embedded within variable annuities.

FIGURE 1

Will the nest egg be enough to fund the "income gap"?



The formula described computes the probability that a given retirement nest egg will be sufficient to finance the "income gap," which is the difference between retirement income needs and pension-like benefits.

The Main Formula: Probability of Retirement Ruin

If a retiree is invested in a standard (balanced) portfolio and plans on withdrawing a fixed inflation-adjusted amount every year during retirement, he or she obviously faces the probability that his or her portfolio will be exhausted while still alive. Whether or not these constant static withdrawals are a realistic assumption for the behavior of actual retirees, the underlying paradigm is the foundation of numerous simulation-based studies. Figure 1 provides a graphical illustration. Nevertheless, under basic portfolio-dynamic assumptions, the formula for this *probability of retirement ruin*—the probability that a fixed spending plan will deplete a retirement nest egg prior to the end of the lifecycle—can be expressed in two distinct steps. First, define:

$$\alpha = \frac{2(\mu + 2\lambda)}{\sigma^2 + \lambda} - 1, \ \beta = \frac{\sigma^2 + \lambda}{2} \qquad 1$$

where μ denotes the retiree's portfolio expected rate of return, σ denotes the volatility or uncertainty surrounding this projected investment return, λ denotes the *mortality rate* of the retiree, and S denotes the inflation-adjusted spending rate as a percentage of the initial portfolio value.

The probability of retirement ruin—1 minus the probability of sustainability—can be obtained via the GammaDist formula in Excel:

$$\text{RetirementRuin} = \text{GammaDist}(S/\beta, \alpha, 1, \text{TRUE}) \qquad 2$$

where the two intermediary variables (α, β) are defined by equation 1 as explicit functions of the four input variables

(μ, σ, λ, S), all of which will be explained in greater detail in a moment. Note that the last expression in the formula (TRUE) is meant to indicate to Excel that the user is interested in the cumulative distribution function (CDF) and not the probability mass function (PMF).

First, let's break equation 1 into bite-sized pieces and translate the Greek into English. The formula depends on four input variables (μ, σ, λ, S) that get mapped into the two intermediate variables (α, β), which then get used in the actual formula.

One can think of the underlying problem in the following way. At retirement the individual has a sum of money (nest egg) that must finance a fixed consumption stream over a random investment horizon. This randomness, of course, comes from the uncertainty of the length of one's life. Note that if the entire nest egg itself were invested in a portfolio that earns a fixed inflation-adjusted return forever, and the horizon were deterministic, one could easily calculate the present value of the desired consumption stream and compare with the retirement nest egg (RNE). This is the textbook example of retirement income planning. If the PV > RNE, the income plan is not sustainable. If the PV < RNE, the income plan is sustainable and the retiree will have enough. But—and this is the critical part—when the retirement horizon is random and the future inflation-adjusted returns themselves are random, the PV is random as well. This then leads to a probability that PV > RNE, which is the probability of retirement ruin or 1 minus the probability of sustainability. Indeed, the above-mentioned formula is akin to computing a z-score from a normal distribution, where alpha is related to the "mean" and beta is related to the "variance" of the distribution. However, the PV of lifetime income is most definitely not normally distributed; it is closer to gamma distributed, hence the GammaDist function.

Here is a numerical example to help develop a better understanding of how to use the formula in general. Assume a generic, 65-year-old client who has just retired with a nest egg of $500,000, which is sitting inside a tax-sheltered plan. Any and all withdrawals from this plan will be taxed at the retiree's marginal tax rate, which is why this paper will not distinguish between the various forms of income such as taxable bonds, tax-efficient equity, etc.

According to mortality tables used by pension actuaries, this typically healthy 65-year-old client has (approximately) a 94% chance of surviving for 5 more years to age 70, a 56% chance of surviving for 20 more years to age 85, and a 16% chance of surviving for 30 more years to age 95. Client retirement horizons are random and they obviously face the *longevity risk* of outliving their nest eggs if they live longer than anticipated. It is hard to overstate the importance of incorporating longevity risk when preparing a retirement plan. The many research studies (and software algorithms) that focus deterministically on 20, 25, or 30 years of retirement are ignoring the most vexing risk of them all.

Either way, the 65-year-old (healthy) client's median remaining lifespan (MRL)—or median number of years until death—is approximately 23 years, which means that there is 50% chance he or she will live beyond age 88. From a mathematical point of view, if there is a 50% chance of living for 23 more years, an *exponential future lifetime* assumption ($e^{-23\lambda} = 0.5$) leads to a mortality rate of $\lambda = 1n[2]/23 = 0.0301$. The average rate of death is approximately 3% per year. Recall that this number is one of the four input variables in equation 1.

Let's further assume that the same 65-year-old would like to withdraw $40,000 per year, pretax in inflation-adjusted terms, during the course of his or her retirement. This represents S = 40/500 = 0.08 or 8% of the initial nest egg and is another one of the four input variables in the retirement ruin formula displayed in equations 1 and 2.

Now, let's emphasize once again up front that it is unrealistic to assume any client will be mechanically adhering to a withdrawal rule of precisely $40,000 per year in inflation-adjusted terms. Every client has the flexibility to modify a plan by spending more or less in any given year. Most likely if the portfolio performs poorly the client will scale back spending and *vice versa* if the portfolio achieves above-average investment performance. However, the point of this exercise is to ascertain whether $40,000 per year withdrawals are sustainable over a random retirement horizon.

In terms of asset allocation, assume the client's investment portfolio is allocated to a mix of balanced mutual funds (or ETFs, SMAs, etc.) and the estimate is that this portfolio will earn $\mu = 0.075$, which is an arithmetic average of 7.5% in any given year, with a standard deviation or volatility of $\sigma = 0.18$, which is 18% per year, both in *inflation-adjusted* terms. These numbers are consistent with the well-known and widely cited numbers by Ibbotson Associates (2004).[4] In some sense, the precise magnitude of these numbers is less important than the general awareness that they can fluctuate and hence are yet another source of retirement risk.

To those who are unfamiliar with investment means and volatilities, what these parameters imply is that in any given year the range of annual investment outcomes is between –10.5% and +25.5% two times out of three, which is one standard deviation away from the mean. For 19 times out of 20 confidence, the range is between –28.5% and +43.5%, which is obviously much wider. Note, also, that all of these numbers are expressed as a continuously compounded rate. Either way, the two numbers 0.075 and 0.18 are the last of the four ingredients needed to use the retirement ruin equation.

The next step is to compute the newly defined *retirement alpha:*

$$\alpha = \frac{2\mu - 4\lambda}{\sigma 2 - \lambda} - 1 = 3.326$$

and the newly defined *beta-adjusted spending* rate:

TABLE 1

Random Life and Returns:
What Is Your Probability of Retirement Ruin?
(Under various portfolio returns (μ) and volatility (σ) combinations, this table displays the probability of retirement ruin—i.e. the retiree's RisQuotient—under a relatively high spending rate of S = 8% and a median remaining life of 23 years.)

Mortality Rate 3.01% | **Median Life (years)** 23.00 | **Retirement Income Spending Rate** 8.00%

	The Volatility of Portfolio's Investment Return				
Expected Return	**5%**	**10%**	**15%**	**20%**	**25%**
1%	82.8%	84.2%	86.6%	89.7%	93.4%
3%	62.8%	66.5%	71.7%	77.5%	83.6%
5%	40.7%	46.4%	54.1%	62.5%	71.1%
7%	22.6%	28.6%	37.2%	47.1%	57.4%
10%	7.0%	11.0%	18.0%	27.3%	38.0%

TABLE 2

Random Life and Returns:
What Is Your Probability of Retirement Ruin?
(Under various portfolio returns (μ) and volatility (σ) combinations, this table displays the probability of retirement ruin—i.e. the retiree's RisQuotient—under a relatively high spending rate of S = 8% and a relatively long median remaining life of 35 years. Quite intuitively, the RisQuotient is high.)

Mortality Rate 1.98% | **Median Life (years)** 35.00 | **Retirement Income Spending Rate** 8.00%

	The Volatility of Portfolio's Investment Return				
Expected Return	**5%**	**10%**	**15%**	**20%**	**25%**
1%	95.7%	95.4%	95.6%	96.7%	98.6%
3%	81.6%	82.8%	85.1%	88.3%	92.0%
5%	57.1%	61.9%	68.1%	74.8%	81.4%
7%	31.5%	39.0%	48.5%	58.4%	67.9%
10%	8.3%	14.1%	23.4%	34.6%	46.4%

$$\frac{S}{\beta} = \frac{2S}{\sigma^2 + \lambda} = 2.558$$

as per the instructions in equation 1. Finally, plugging these values into Excel as per equation 2, *GammaDist(2.558,3.326,1,TRUE)* results in the number 39.3%, which is the probability of retirement ruin. Intuitively, a higher retirement alpha will result in a lower risk of retirement ruin, while a higher beta-adjusted spending rate will lead to a higher retirement ruin, or Retirement RisQuotient.[5]

Now, for example, if this retiree was planning on spending only $35,000 per year of retirement (in inflation-adjusted terms), the spending rate would fall to 7% of the initial $500,000 and the beta-adjusted spending rate would be reduced to 2.239 instead of the earlier 2.558 value. It is therefore no surprise that when this value is plugged into the same function *GammaDist(2.239,3.326,1,TRUE)*, the probability of retirement ruin is reduced to 31.3% from 39.3%. A lower beta-adjusted level of spending results in a lower ruin probability. Likewise, if instead of assuming that the portfolio will earn 7.5% in any given year, one is more optimistic and uses $\mu = 0.09$, which is 1.5% per annum above the earlier number, the retirement alpha variable becomes 3.806 instead of the earlier 3.326 value. And, under the earlier $40,000 spending rate and beta-

TABLE 3

Random Life and Returns:
What Is Your Probability of Retirement Ruin?

(Under various portfolio returns (μ) and volatility (σ) combinations, this table displays the probability of retirement ruin—i.e. the retiree's RisQuotient—under a relatively high spending rate of S = 4% and a relatively long median remaining life of 35 years. Quite intuitively, the RisQuotient is high.)

Mortality Rate 1.98% | **Median Life (years)** 35.00 | **Retirement Income Spending Rate** 4.00%

	The Volatility of Portfolio's Investment Return				
Expected Return	**5%**	**10%**	**15%**	**20%**	**25%**
1%	60.0%	67.0%	75.8%	85.2%	94.4%
3%	25.2%	34.8%	47.8%	61.8%	75.7%
5%	7.0%	13.3%	24.4%	38.7%	54.4%
7%	1.4%	3.9%	10.4%	21.3%	35.7%
10%	0.1%	0.4%	2.1%	7.1%	16.3%

TABLE 4

Random Life and Returns:
What Is Your Probability of Retirement Ruin?

(Under various portfolio returns (μ) and volatility (σ) combinations, this table displays the probability of retirement ruin—i.e. the retiree's RisQuotient—under a relatively high spending rate of S = 4% and a relatively low median remaining life of 23 years. Quite intuitively, the RisQuotients are lower compared to the previous three tables.)

Mortality Rate 3.01% | **Median Life (years)** 23.00 | **Retirement Income Spending Rate** 4.00%

	The Volatility of Portfolio's Investment Return				
Expected Return	**5%**	**10%**	**15%**	**20%**	**25%**
1%	37.1%	44.8%	55.7%	67.9%	80.3%
3%	15.3%	21.9%	32.4%	45.7%	60.2%
5%	5.0%	8.8%	16.3%	27.5%	41.5%
7%	1.3%	3.0%	7.2%	15.1%	26.5%
10%	0.1%	0.5%	1.7%	5.2%	12.1%

adjusted spending rate of 2.558, we arrive at a retirement ruin probability, using the same formula *GammaDist(2.558,3.806,1,TRUE)* of 29.1% instead of 39.3%. This is an improvement, i.e. a reduction of risk, of close to 10 percentage points. Once again, a higher retirement alpha value will reduce the ruin probability.

Tables 1–4 provide a range of numbers based on equations 1 and 2 assuming differing spending rates, mortality rates, and portfolio characteristics. The four tables correspond with four extremes. Table 1 represents a relatively high spending rate of 8% (i.e. annual $8 withdrawals per $100 nest egg) of a typical retiree with a 23-year median remaining lifespan. Table 2 illustrates the results for the same 8% spending rate, but for a much younger (early) retiree facing a median remaining lifespan of 35 years. Tables 3 and 4 examine the same mortality rates but under a much lower spending rate of 4% of the initial nest egg. All of these rates are adjusted for inflation.

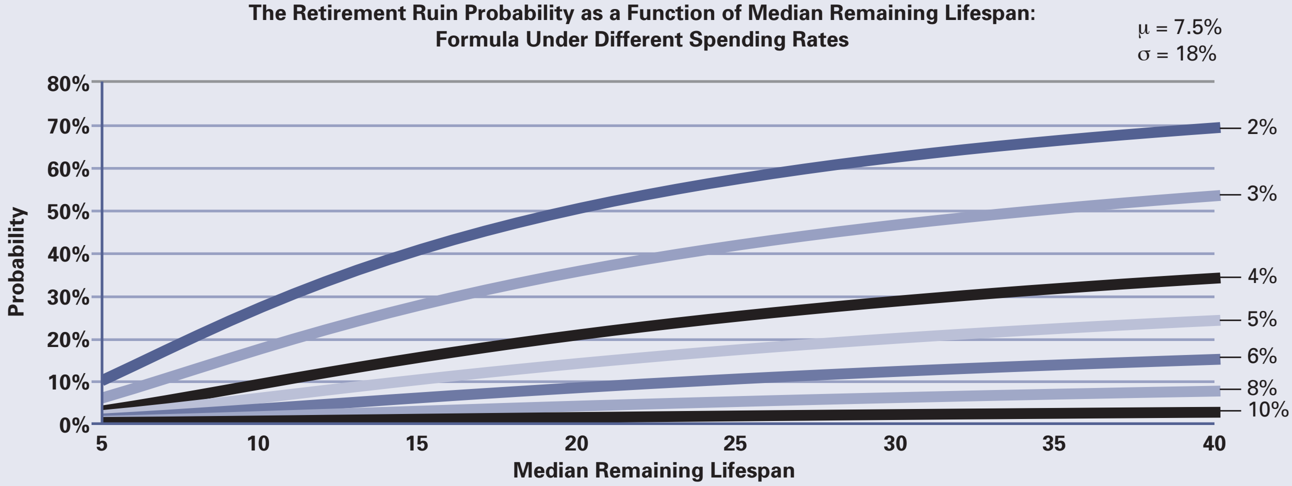
Thus, for example, a portfolio that is expected to earn 5% per annum with a volatility of 10% will result in a 46.4% ruin probability under an 8% spending rate and 23-year median remaining lifespan, but only an 8.8.% probability of ruin under a 4% spending rate. Halving the spending rate reduces the ruin by over 38 percentage points. It is often stated that spending money in retirement is akin to creating your own pseudo "bear market" since each year the withdrawal process reduces portfolio growth by the spending rate S. Notice that, consistent with this heuristic, increasing S from 4 to 8% under a 7% expected investment return and a 20% volatility has the same impact as reducing the expected return from 7 to 3% under an 8% spending rate.

The important take-away is as follows. This paper does not advocate a particular spending/withdrawal rate or equity/bond allocation as being optimal or prudent. In fact, it is doubtful that any spending rate will be adhered to on a constant basis. Rather, the answers to all these important questions are intertwined and depend on the level of risk aversion (or tolerance) of the retiree, compared against the Retirement RisQuotient of the plan.

The point is that spending rates as high as 8–10% can be defended if the forward-looking volatility is low enough and the forward-looking expected return is high enough. One important take-away is that the retirement income debate should be phrased in terms of the anticipated and forward-looking equity risk premium (ERP) as opposed to the historical performance of these asset classes. The formulas in equations 1 and 2 provide this link.

FIGURE 2

The Retirement Ruin Probability as a Function of Median Remaining Lifespan: Formula Under Different Spending Rates



Note: The mortality rate λ, which is needed for the main formula, is obtained by dividing the natural logarithm of 2, which is approximately 0.693, by the median remaining lifespan (MRL). Thus, for example, when the MRL is 25 years, the relevant value of $\lambda = 2.772\%$.

Figure 2 provides a graphical illustration of the retirement ruin probabilities under the same 7.5% expected return and 18% volatility assumption, for a variety of spending rates and at various retirement ages. The differing retirement ages are captured by changing the MRL. At the extreme left of the figure there is a 50% chance of living for five more years and at the extreme right there is a 50% chance of living for 40 more years. Remember that the MRL is used to compute the mortality rate (λ) which is at the heart of equation 1.

As one would expect, the greater the spending rate—which ranges from a low of 2% to a high of 10%—the higher the probability of retirement ruin. Likewise, the longer the MRL, the greater is the probability of ruin. Notice that when planning on spending 10% of the initial nest egg each and every year of retirement, the probability of ruin ranges from a low of about 10% to a high of 69% depending on how old one is when one starts retirement spending. In contrast, when planning on spending only 2% of the nest egg during retirement, the ruin probability ranges from a minimal 0.1% to 2.7% depending on age.

Another possible way of interpreting or thinking about the MRL is to treat this number as the 50% mark for a couple's retirement plan. If the couple expects at least one member to be alive for at least 30 years, they should focus on numbers to the right of 30 on the x axis. Those who can tolerate living on the 30% ruin fault line can obviously spend much more than those who prefer a southern climate in the 10% ruin region.

On the Back of an Envelope: No Software Required

It is possible to use and apply the main equations 1 or 2 without having access to Microsoft Excel, or any other statistical software package that does calculus. In fact, using the two key variables of *retirement alpha* (α) and the *beta-adjusted spending* rate (S/β), also called the retirement beta, one can calculate the retirement ruin probability on the back of a (large) envelope. Table 5 provides the main exhibit.

TABLE 5

Back-of-the-Envelope Calculation of Retirement Ruin Risk

Retirement Alpha	Beta-Adjusted Spending Level																
	0.5	0.65	0.8	0.95	1.1	1.25	1.4	1.55	1.7	1.85	2	2.15	2.3	2.45	2.6	2.75	2.9
4.50	0.1[a]	0.2	0.4	0.7	1.2	1.9	2.8	4.0	5.4	7.0	8.9	10.9	13.2	15.7	18.3	21.1	24.0
4.30	0.1	0.2	0.5	1.0	1.6	2.5	3.7	5.1	6.7	8.6	10.8	13.1	15.7	18.4	21.3	24.3	27.4
4.10	0.1	0.4	0.8	1.4	2.23	3.3	4.7	6.4	8.4	10.6	13.0	15.7	18.5	21.5	24.6	27.8	31.1
3.90	0.2	0.5	1.1	1.9	3.0	4.4	6.1	8.1	10.3	12.9	15.6	18.6	21.7	24.9	28.3	31.6	35.0
3.70	0.3	0.8	1.5	2.6	4.0	5.7	7.7	10.1	12.7	15.6	18.6	21.9	25.2	28.7	32.2	35.7	39.2
3.50	0.5	1.2	2.1	3.5	5.2	7.3	9.7	12.4	15.4	18.6	22.0	25.5	29.1	32.8	36.4	40.1	43.7
3.30	0.8	1.7	3.0	4.7	6.8	9.3	12.1	15.3	18.6	22.2	25.8	29.6	33.4	37.2	41.0	44.7	48.3
3.10	1.2	2.4	4.1	6.2	8.8	11.7	15.0	18.5	22.3	26.1	30.1	34.0	38.0	41.9	45.7	49.4	53.0
2.90	1.8	3.4	5.5	8.2	11.3	14.7	18.4	22.3	26.4	30.5	34.7	38.8	42.9	46.8	50.6	54.3	57.8
2.70	2.6	4.7	7.4	10.6	14.3	18.2	22.4	26.7	31.0	35.4	39.7	43.9	48.0	51.9	55.7	59.3	62.6
2.50	3.7	6.5	9.9	13.7	17.9	22.4	26.9	31.5	36.1	40.7	45.1	49.3	53.3	57.2	60.8	64.2	67.4
2.30	5.4	8.9	13.0	17.5	22.2	27.1	32.1	36.9	41.7	46.3	50.7	54.8	58.8	62.5	65.9	69.1	72.0
2.10	7.6	12.0	16.9	22.0	27.3	32.6	37.8	42.8	47.6	52.2	56.5	60.5	64.2	67.7	70.8	73.8	76.4
1.90	10.6	16.0	21.6	27.4	33.1	38.7	44.0	49.1	53.8	58.2	62.3	66.1	69.6	72.7	75.6	78.2	80.6
1.70	14.7	21.0	27.3	33.6	39.6	45.3	50.7	55.6	60.2	64.4	68.2	71.6	74.8	77.6	80.1	82.4	84.4
1.50	19.9	27.1	34.1	40.7	46.8	52.5	57.7	62.4	66.6	70.4	73.9	76.9	79.6	82.1	84.2	86.1	87.8
1.30	26.5	34.4	41.8	48.5	54.4	60.0	64.8	69.1	72.9	76.3	79.2	81.8	84.1	86.2	87.9	89.5	90.8
1.10	34.7	43.1	50.5	56.9	62.6	67.6	71.9	75.6	78.9	81.7	84.2	86.3	88.2	89.8	91.2	92.4	93.4

[a] All numbers are percentages.

The columns of Table 5 represent increasing levels of beta-adjusted spending (retirement beta), while the rows capture decreasing levels of retirement alpha. From any given cell in the table, moving down and/or to the right will increase the probability of retirement ruin. Quite naturally, when spending more or generating a smaller retirement alpha, the probability of ruin and Retirement RisQuotient is higher. Within this table the numbers range from a low of almost zero to a high of 93%. Ideally one wants to position oneself in the upper left-hand corner of this two dimensional risk map.

For example, using the 8% spending rate numbers from the previous section, when the inflation-adjusted portfolio mean return was 7.5%, the return volatility was 18%, and mortality rate was 3.01%, the value of the retirement alpha was 3.326 and the beta-adjusted spending was 2.558. This led to a probability of retirement ruin of 39.3%. Now, if we examine Table 5, the closest value for α is 3.3 and the closest value for S/β is somewhere between 2.45 and 2.6, with equal distance. This leads to a probability of 37–41%, which closely approaches the true 39.3% probability. Likewise, if the spending is reduced from 8% of the initial nest egg to 4% of the nest egg, the retirement alpha value remains unchanged but the beta-adjusted spending rate declines to 1.279 from 2.558. The exact retirement ruin probability is now 9.5% according to equation 1. In Table 5, when the beta-adjusted spending rate falls between 1.25 and 1.4, the ruin probability is between 9.3% and 12.1%.

Those who decide to use such a table instead of the precise formula occasionally might encounter portfolio return variables μ, σ and/or mortality rates λ and/or inflation-adjusted spending rates S that will induce α, S/β values lying beyond the ranges listed in Table 5. In this case, the direction via which one leaves the table should provide a general picture of sustainability. If, for example, the computed retirement alpha value is very low (<1), this is bad news. The ruin probability will be high, and so is a very large (>3) level of your beta-adjusted spending rate. Of course, if one wants a more precise answer compared to what the table's heuristics can provide, then fire up Excel.

In sum, the way to use this table is in two stages. First, estimate the four main variables (μ, σ, λ, S) and convert those into the key variables (α, S/β), based on equation 1. Then, take these two summary numbers to Table 5 and look up the retirement ruin probability based on rows and columns. Obviously, this exercise will not produce the precise number generated by equations 1 and 2 since they are approximating the row and column variable to within two digits, but the results are quite close. In fact, given that this entire exercise hinges on a number of embedded assumptions described in the next section, it is prudent to avoid emphasizing numerical results beyond the first two digits of Table 5.

Using this approach, one can envision a simple questionnaire that clients (users) can complete to measure their Retirement RisQuotient. Soon-to-be retirees would be asked about their gender, age, family history, and health information to arrive at an estimate of their MRL which leads to their implied mortality rate (λ). The same user would then specify an inflation-adjusted retirement income spending rate net of any defined-benefit (DB) pension income, preexisting life annuities, and other longevity-insured income stream, which would lead to an estimate for S. Finally, they would be asked about the specific composition of their investment portfolio that would lead to estimates of μ, σ. All of these together would be fed into a formula or Table 5 that would generate a Retirement RisQuotient. The higher this number the lower the sustainability of the retirement plan.

Caveats and Warnings for the Quants: Know the Limits

For those readers who want to know where this formula comes from, or how accurate it is relative to extensive Monte Carlo simulations, one is encouraged to consult the book by Milevsky (2006)[6] or the paper by Milevsky and Robinson (2005).[7] Both references contain much greater mathematical detail relative to what can be described in a brief article such as this. In a nutshell, however, there are a number of important assumptions upon which equations 1 and 2 are based. It is important to spell them out explicitly.

First, the formula assumes that the underlying investment portfolio value obeys a geometric Brownian motion (GBM). This means that in any given year the one-plus-investment return is lognormally distributed and the logarithm of the one-plus-investment return (the continuously compounded return) is normally distributed. And, while this embedded assumption has been used by thousands of practitioners ever since the pioneering days of Markowitz and Merton, it has been questioned by thousands of academics as well. In GBM's defense, it seems the brunt of the criticism has been leveled at the inability of the normal distribution to capture the rare and infrequent tails in market returns over short horizons, as opposed to the (very) long term that would be relevant for the sustainability of retirement portfolios. Thus, the author feels comfortable using the normal distribution within the calculus of retirement income, although it is recommended to use a portfolio volatility estimate (σ) that is slightly higher than historical estimates to take account of the above-mentioned model risk.

Second, the model is predicated on the uncertain length of human life being *exponentially distributed.* This implies that the mortality rate is constant over time and that the probability of living for T more years is precisely $e^{-\lambda T}$, where λ denotes the instantaneous mortality rate. Indeed, many insurance actuaries will recoil in horror from this assumption knowing that the only organism on earth with an exponential remaining lifespan is a lobster! Human mortality rates increase with age and certainly do not remain constant over time. Yet, interestingly enough, when equations 1 and 2 are calibrated to the actuarially correct median lifespan, the results are remarkably accurate when compared against the true ruin probability under the complete mortality rates.

Third, even under an exponential remaining lifespan and lognormally distributed investment returns, equation 1 is an approximation based on moment matching techniques. In other words, the ruin probability is accurate in the limit and only when $\lambda \rightarrow 0$ does the formula converge to the truth. Once again, this approximation has been stress tested quite extensively in a number of other papers (see Milevsky and Robinson for example)[8], and the results indicate errors that are less than 5% in the most extreme cases.

Finally, notwithstanding the above-mentioned caveats, there are a number of important limits to the range of equation 1. For example, the retirement alpha value must be greater than zero, or the underlying integral "blows up" and the formula produces errors. This places restrictions on the *geometric mean*[9] investment return ($\mu - 0.5\sigma^2$) relative to the mortality rate (λ). Specifically, one must obey the relationship $\mu - 0.5\sigma^2 + 1.5\lambda > 0$ in order to satisfy $\alpha > 0$. Thus, when using the formula for any retirement age and any mortality rate, a sufficient condition is that the geometric mean investment return is positive. Notice the critical role of the geometric mean investment return $\mu - 0.5\sigma^2$ relative to the arithmetic mean investment return μ in determining the range of sustainable spending rates. Not only is the difference between the two critical; investment volatility matters.

Engineering Sustainable Income: The Role of Product Allocation

In addition to the traditional menu of asset classes such as stocks, bonds, and cash that are available to retirees, there are two important product classes that can further reduce the probability of retirement ruin and increase the sustainability of a retirement plan. The first product class is a life annuity (payout or income annuity) and the second product class consists of exchange-traded put options used to protect the portfolio. And, while many retirees and their advisors might shun exchange-traded options, the closest retail substitute is a variable annuity—manufactured by insurance companies—that contains guaranteed minimum withdrawal benefits (GMWB) and guaranteed minimum income benefits (GMIB). These products effectively create downside protection in the critical early years of retirement, sometimes labeled the retirement risk zone or retirement red zone.

Buying a life annuity that provides lifetime income can greatly improve sustainability. This point was made by Ameriks, Veres and Warshawsky[10] using simulation arguments, or by Chen and Milevsky[11] using utility-based arguments; the same insight can also be extracted from equations 1 and 2 in a number of ways. First, by receiving lifetime income from the insurance company the retiree is exposed to a lower level of longevity risk. This reduces the probability of retirement ruin. More importantly, annuitizing a percentage of one's nest egg generates a mortality subsidy, which increases the portfolio's investment return. This concept has been explained in Milevsky (2005),[12] but can be illustrated using a simple story.

Imagine that a large group of retirees of exactly the same age—and each subject to the same mortality rate

(λ)—pool their retirement nest egg into one large portfolio whose value is denoted by the symbol W_t. The members of the pension portfolio invest their collective wealth (W_t) in a mutual fund that is expected to earn μ in any given year. Every day the members of this pension fund, who are still alive, withdraw the exact same (S/365)W, stated as a percentage of the original nest egg. However, those who do not survive forfeit their retirement assets to the group. In other words, these funds are not bequeathed to an estate or beneficiary. Now, since this pool of money is being augmented by the assets of the deceased, the money will last longer compared to the situation in which each retiree managed his or her own portfolio independently. In fact, over a short period of time the portfolio will increase by $W(\mu + \lambda)$, which is the sum of the expected investment return and the mortality rate.

Here is another way to justify this issue. So far this entire article has emphasized real (inflation-adjusted) spending. Thus, if one is financing retirement from a basic mutual fund, then conceptually each year the systematic withdrawal plan (SWiP) is adjusted to account for a realized inflation rate. Indeed, one could create the same stable and predictable income stream by purchasing a life annuity from an insurance company and hence completely hedge the longevity risk. However—and this is key—there are very few companies that offer a competitively priced inflation-adjusted life annuity. More importantly, the rate by which one might adjust annual spending would depend on a personal inflation rate, which is unlikely to ever be available from an insurance company.[13] This is why we assume the available annuity is of a variable (tontine) type in which payments fluctuate based on the performance of the stock market, but withdrawals are in constant (personal) inflation-adjusted terms.

From a practical perspective, this implies that we can use the exact same equations 1 and 2 to compute the probability of retirement ruin, except that we substitute the value of $\mu + \lambda$ instead of μ. Clearly this will reduce the risk and increase the sustainability of the portfolio. And, although real-world annuities have additional features that are not quite captured by this framework, the underlying principle is exactly the same.

Table 6 provides some numerical indication of how annuitization will reduce the probability of retirement ruin, i.e. the risk quotient. For example, if a 65-year-old retiree plans on spending 8% (in real terms) and is anticipating a median age at death of 83.9, then the implied mortality rate is $\lambda = 1n[2]/(83.9–65)=0.03667$. The nonannuitized probability of retirement ruin is 41.15%, under portfolio parameters of 7% expected return and 20% volatility. However, if this person completely annuitizes his or her nest egg, which is obviously the extreme case, the instantaneous expected return will increase from

TABLE 6

You Retired and Are Planning on Spending 8% of Your Initial Nest Egg per Year; Adjusted for Inflation
Investment Portfolio Is Expected to Earn 7% with a Volatility of 20%
(These assume variable payout annuities with no management or insurance fees.)

Retirement Age	Median Age at Death	Mortality Rate (%)	Ruin Probability with Life Annuities (%)	Ruin Probability without Life Annuities (%)
50	78.1	2.467	34.38	52.80
55	83.0	2.476	34.25	52.71
60	83.4	2.962	27.81	47.63
65	83.9	3.667	20.55	41.15
70	84.6	4.748	13.05	33.02
75	85.7	6.478	6.55	23.61
80	87.4	9.367	2.35	14.22

Note: Using part of your retirement portfolio to purchase variable life (e.g. payout or income) annuities will reduce the retirement ruin or risk quotient. This is achieved by effectively increasing the portfolio's return by the implied longevity yield or the mortality credits contained within the payout annuity.

7% to 10.3667% due to the mortality credits. In other words, since retirees are dying and abandoning their wealth at a rate of 3.667% per year, the portfolio will grow by an extra 3.667% per year. If we then compute the retirement ruin probability using the exact same λ = 0.03667, σ = 0.20, S = 0.08 variables, but using 10.366% instead of 7% for the input variable μ, one is left with a retirement ruin probability of 20.55%. Hence, life annuities in their most general form will reduce the retirement ruin probability and increase sustainability. And, while we have ignored management and insurance fees, which can easily be incorporated by reducing the magnitude of $\mu + \lambda$, the underlying message remains the same.

Just as important as the life annuity product class, downside protection (e.g., portfolio insurance or put options) also reduces the retirement ruin probability. By downside protection I mean having access to derivative products that limit the range of investment outcomes and thus eliminating the "bad tails." These types of guarantees can be purchased in the open market on option exchanges, or the process can be outsourced to intermediaries such as insurance companies that offer withdrawal benefits on variable annuities. All of these GMWBs, GMIBs, and guaranteed minimum accumulation benefits (GMABs) effectively reduce the magnitude of the volatility variable σ, which in turn reduces the retirement ruin probability *assuming the insurance fees charged for this protection are not too high.*

For example, look again at Table 1. In this case a portfolio that is expected to earn 7% with a volatility of 20% will result in a 47.1% ruin probability under an 8% withdrawal rate, for a retiree planning for a median lifespan of 23 years. Note that if the volatility of the portfolio's return can be reduced from 20% to 10%, the probability of retirement ruin will (obviously) be reduced from 47.1% to 28.6%, which is substantial. This reduction in volatility would virtually eliminate the possibility of any large and pleasant (upside) surprises in exchange for avoiding large unpleasant (downside) surprises. And, even if the retiree has to pay an extra (abnormally high) fee of 200 basis points, which reduces the expected return from 7% to 5%, the probability of retirement ruin still declines from 47.1% to 46.4%. In sum, the main formulas 1 and 2 illustrate how two important product classes—life annuities and downside protection—impact the probability of retirement ruin, all in one parsimonious framework.

Conclusion: Manage Your Risk

Simple mathematical formulas in finance can often seduce their users into a false sense of precision regarding the true behavior of the real world. Billions of dollars have been lost by financial scientists who believed markets obeyed formulas. And, while academics might occasionally fall prey to the beauty of formulas, seasoned practitioners know the limits of their relevance. Nevertheless, the author believes this article contains a number of qualitative insights that *do apply* to real people facing retirement.

First and quite obviously, starting retirement early generates a higher Retirement RisQuotient, all else being equal. Likewise, greater retirement spending rates, lower portfolio returns, and higher investment volatility all increase this risk metric. These insights will not come as news to the financial planning profession. However, this framework has also provided a rigorous justification for using life annuities and other riders associated with variable immediate and income annuities in one parsimonious framework, since both these products increase the sustainability of a retirement income portfolio and reduce the Retirement RisQuotient. ■

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(1) The Retirement RisQuotient is a trademark of The QWeMA Group and the author.

(2) This is very different from the rules and terminology governing required minimum distributions (RMDs) from qualified plans, where a particular spending rate is stated in terms of a percentage of an account value at the beginning or end of a given year.

(3) W.P. Bengen, "Determining Withdrawal Rates Using Historical Data," *Journal of Financial Planning* 7 (October 1994):171–181; K. Ho, M. Milevsky, and C. Robinson, "How to Avoid Outliving Your Money," *Canadian Investment Review* 7 (Fall 1994): 35–38; P.L. Cooley, C.M. Hubbard,

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(4) Ibbotson Associates, *Stocks, Bonds, Bills and Inflation: 2004 Yearbook* (Chicago, IL: Ibbotson Associates).

(5) Note that the formula in Excel can also be typed in as GammaDist(S,alpha,beta, TRUE), where instead of the value 1 in the third argument of equation (2) the formula uses beta, but at the same time the spending rate S is not adjusted for beta. The result is the same since mathematically GammaDist (a,b,c,TRUE) is the same as GammaDist (a/c,b,1, TRUE). Other related papers in the literature have used the term beta to describe S/beta.

(6) M.A. Milevsky, *The Calculus of Retirement Income: Financial Models for Pensions and Insurance* (Cambridge, U.K.: Cambridge University Press, 2006).

(7) M.A. Milevsky and C. Robinson, "A Sustainable Spending Rate without Simulation," *Financial Analysts Journal* 64 (Winter 2005).

(8) Ibid.

(9) When the investment return is assumed lognormally distributed, the gap between the (larger) arithmetic mean and the (smaller) geometric mean is one-half the volatility squared. Note that all parameters assume continuous compounding.

(10) J. Ameriks, M. Veres, and M. Warshawsky, "Making Retirement Income Last a Lifetime," *Journal of Financial Planning* (December 2001): 60–76.

(11) P. Chen and M.A. Milevsky, "Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities," *Journal of Financial Planning* (2003): 64-72.

(12) Milevsky, "The Implied Longevity Yield" (2005).

(13) For a more detailed discussion of personal inflation rates as they pertain to retirement income, see M.A. Milevsky, "Retirement Income University," *Research Magazine* (May 2007).

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