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A Sustainable Spending Rate without Simulation

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Financial commentators have called for more research on sustainable spending rates for individuals and endowments holding diversified portfolios. We present a forward-looking framework for analyzing spending rates and introduce a simple measure, stochastic present value, that parsimoniously meshes investment risk and return, mortality estimates, and spending rates without resorting to opaque Monte Carlo simulations. Applying it with reasonable estimates of future returns, we find payout ratios should be lower than those many advisors recommend. The proposed method helps analysts advise their clients how much they can consume from their savings, whether they can retire early, and how to allocate their assets.

"Retirees Don't Have to Be So Frugal: Here Is a Case for Withdrawing Up to 6 Percent a Year . . ."

Jonathan Clements
Wall Street Journal (17 November 2004)

Retirees and endowment and foundation trustees share a common dilemma: How much can we spend without running out of money during our lifetime? Sustainable withdrawal and spending rates have been the subject of sporadic academic research over the years. The issue has developed new urgency, however, as the wave of North American Baby Boomers approaches retirement and seeks guidance on "what's next" for their retirement savings. Complicating the issue is the likelihood that Baby Boomers can expect to live for a long (but random) time in retirement as medical advances stretch human lifetimes.

For endowments and foundations, this topic has a 30-year history going back to a special session at the American Economics Association devoted to spending rates, in which Tobin (1974) cautioned against consuming anything other than dividends and interest income.[1] Also in the 1970s, Ennis and Williamson (1976) analyzed appropriate asset allocation in conjunction with a given spending policy. More recently, Altschuler (2000) argued that endowments are actually "too stingy" and are not spending enough; Dybvig (1999) discussed how a pseudo portfolio insurance scheme used in asset allocation can protect a desired level of spending; and Hannon and Hammond (2003) discussed the impact of the recent (poor) market performance on the ability to sustain payouts.

In the parallel retirement planning arena, Bengen (1994), Ho, Milevsky, and Robinson (1994), Cooley, Hubbard, and Walz (1998, 2003), Pye (2000), Ameriks, Veres, and Warshawsky (2001), and Guyton (2004) have run financial experiments incorporating historical, simulated, and scrambled returns to quantify the sustainability of various *ad hoc* spending policies and consumption rates for retired individuals. These results usually advocated withdrawals in the 4–6 percent range of initial capital depending on age and asset allocation and then increasing at the rate of inflation and/or contingent on market performance.

The problems with the growing number of these and similar studies based on Monte Carlo simulations—which are intellectually motivated by the "game of life" simulations envisioned by Markowitz (1991)—are that they (1) are difficult to replicate, (2) conduct only a minimal number of simulations, and (3) provide little pedagogical intuition on the financial trade-off between retirement risk and return.[2]

Arnott (2004, p. 6) claimed that "our industry pays scant attention to the concept of sustainable spending, which is key to effective strategic planning for corporate pensions, public pensions, foundations, and endowments—even for individuals." Financial advisors continue to test the sustainability of spending strategies, but the financial literature lacks a coherent modeling framework on which to base the discussion.

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We provide an intuitive and consistent planning model by deriving an analytic relationship between spending, aging, and sustainability in a random portfolio environment. We introduce the concept of stochastic present value (SPV) and an expression for the probability that an initial corpus or investment (nest egg) will be depleted under a fixed consumption rule when both rates of return and time until death are stochastic. And, in contrast to almost all other authors who have tackled this problem, we do not depend on Monte Carlo simulations or historical (bootstrap) studies. Instead, we base the analysis on the SPV and a continuous-time approximation under lognormal returns and exponential lifetimes.

In the case of a foundation or endowment with an infinite horizon (perpetual consumption), this formula is exact. In the case of a random finite future lifetime (the situation of a retiree), the formula is based on moment-matching approximations, which target the first and second moments of the "true" stochastic present value. The results are remarkably accurate when compared with more costly and time-consuming simulations.

We provide numerical examples to demonstrate the versatility of the closed-form expression for the SPV in determining sustainable withdrawal rates and their respective probabilities. This formula, which can easily be implemented in Excel, produces results that are within the standard error of extensive Monte Carlo simulations.[3]

The Retirement Finances Triangle

The main qualitative contribution of this article can be understood by reference to the triangle in **Figure 1**. It provides a graphical illustration of the relationships among the three most important factors in retirement planning: spending rates, investment asset allocation, and mortality (determined by gender and age). We link these three factors in one parsimonious manner by using the "probability of retirement ruin," where "ruin" is defined as outliving one's resources, as a risk metric to gauge the relative impact of each factor and trade-offs between them.

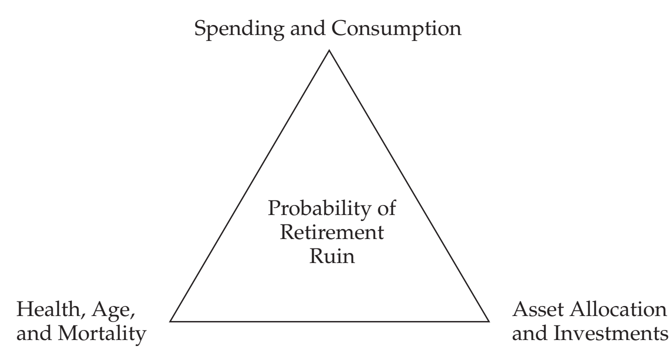
We evaluate the stochastic present value of a given spending plan at a given age under a given portfolio allocation at the initial level of wealth to determine the probability that the plan is sustainable. Increasing the age of the retiree at retirement, reducing the spending rate, or increasing the portfolio return will each shift the mass of the SPV closer to zero (which means reducing the area inside the triangle) and thus generate a higher probability that an initial nest egg will be enough to sustain the plan. Reducing the age at retirement, increasing the spending rate, or reducing the portfolio return will shift the mass of the SPV away from zero—increasing the area inside the triangle—and will thus increase the probability of ruin.

Stochastic Present Value of Spending

The SPV concept is borrowed from actuaries in the insurance industry, who use a similar idea to compute the distribution of the present value of mortality-contingent liabilities, such as pension annuities and life insurance policies. At any given time, an insurance company is thus able to quantify the amount of reserves needed today to fulfill all future liabilities with 99 percent or 95 percent certainty.

The same idea can be applied to retirement planning. Retirees can use an SPV model to compute the size of the retirement portfolio they need in order to draw down a specified annual amount while not incurring more than a specified probability of running out of money during their lifetime.

Figure 1. Retirement Finances Triangle



In the language of stochastic calculus, the probability that a diffusion process that starts at a value of w will hit zero prior to an independent "killing time" can be represented as the probability that a suitably defined SPV is greater than the same w. Imagine that you invest a lump sum of money in a portfolio earning a real (after-inflation) rate of return of R percent a year and you plan to consume/spend a fixed real dollar each and every year until some horizon denoted by T. If the horizon and investment rate of return are certain, the present value (PV) of your consumption at initial time zero, t_0, is

$$PV = \sum_{i=1}^{T} \frac{1}{(1+R)^i} = \frac{1-(1+R)^{-T}}{R}, \tag{1}$$

which is the textbook formula for an *ordinary simple annuity* of \$1. In a deterministic world, if you start retirement with a nest egg greater than the PV in Equation 1 times your desired annual consumption, your money will last for the rest of your T-year life. If you have less than this amount, you will be "ruined" at some age prior to death.

For example, if R = 7 percent and T = 25, the required nest egg is 11.65 (the PV in Equation 1) times your real consumption. If you have more than this lump sum of wealth at retirement, your plans are sustainable. If you start your retirement years with 10 times your desired real annual consumption, then you will run out of money in 17.79 years. Note that as T goes to infinity, which is the endowment case, the PV converges to the number $1/R$. At $R = 0.07$, the resulting PV is 14.28 times the desired consumption.

Human beings have an unknown life span, and retirement planning should account for this uncertainty. **Table 1** illustrates the probabilities of survival based on mortality tables from the U.S.-based Society of Actuaries. For example, a 65-year-old female has a 34.8 percent chance of living to age 90; a 65-year-old male has a 23.7 percent chance of living to age 90. Although the oft-quoted statistic for life expectancy is somewhere between 78 and 82 years in the United States, this statistic is relevant only at the time of birth. If pensioners reach their retirement years, they may be facing 25–30 more years of life with substantial probability because conditional life expectancy increases with age.

Should a 65-year-old plan for the 75th percentile or 95th percentile of the end of the mortality table? What T value should be used in Equation 1? The same questions apply to investment return R. The average real investment returns from a broadly diversified portfolio of U.S. equity during the past 75 years have been in the vicinity of 6–9 percent, according to Ibbotson Associates (2004), but the year-by-year numbers can vary widely. So, again, what number should be used in Equation 1?

Table 1. Conditional Probability of Survival at Age 65

To Age:	Female	Male
70	93.9%	92.2%
75	85.0	81.3
80	72.3	65.9
85	55.8	45.5
90	34.8	23.7
95	15.6	7.7
100	5.0	1.4

Source: Society of Actuaries RP-2000 Table (with projection).

The aim is not to guess or take point estimates but, rather, to actually account for this uncertainty within the model itself. In a lecture at Stanford University, Nobel Laureate William F. Sharpe amusingly called the (misleading) approach that uses fixed returns and fixed dates of death "financial planning in fantasyland."

So, in contrast to the deterministic case—in which both the horizon and the investment return are certain—when both of these variables are stochastic, the analog to Equation 1 is *stochastic present value*, defined as

$$SPV = \frac{1}{(1+\tilde{R}_1)} + \frac{1}{(1+\tilde{R}_1)(1+\tilde{R}_2)} + \ldots + \frac{1}{\prod_{j=1}^{\tilde{T}}(1+\tilde{R}_j)} = \sum_{i=1}^{\tilde{T}} \prod_{j=1}^{i} (1+\tilde{R}_j)^{-1}, \tag{2a}$$

where the new variable $\tilde{T}$ denotes the random time of death (in years) and the new variable $\tilde{R}_j$ denotes the random investment return during year j. (For the infinitely lived endowment or foundation, $\tilde{T} = \infty$.)

The intuition behind Equation 2a is as follows. Looking forward, a retiree must sum up a random number of terms, in which each denominator is also random. The first item discounts the first year of consumption at the first year's random investment return. The second item discounts the second year's consumption (if the individual is still alive) at the product (the compounded rate) of the first and second years' random investment return. And so on.

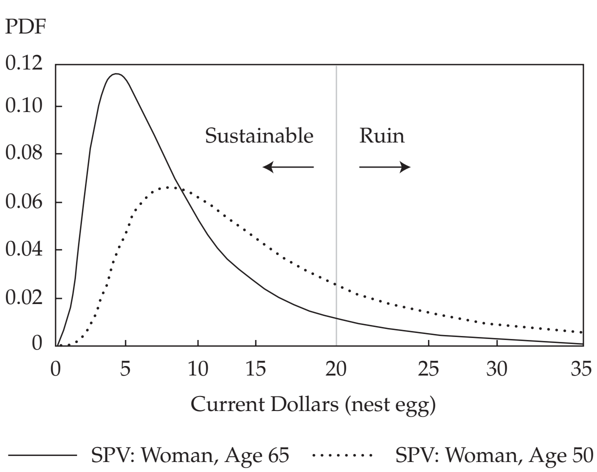
If the investment return frequency is infinitesimal, the summation sign in Equation 2a converges to an integral and the product sign is converted into a continuous-time diffusion process. The continuous-time analog of Equation 2a can be written as follows:

$$SPV = \int_0^{\infty} \text{prob}\left(\tilde{T} > t\right) R_t^{-1} dt, \qquad (2b)$$

where R_t denotes the *total* cumulative investment return (which is random) from the initial time, t_0, until time t. The exponent of –1 discounts the $1, conditional on survival, back to t_0.

The SPV defined by either Equation 2a in discrete time or Equation 2b in continuous time can be visualized as shown in **Figure 2**. The stochastic present value is a random variable with a probability density function (PDF) that depends on the risk–return parameters of the underlying investment-generating process and the random future lifetime. The *x*-axis of Figure 2 is initial wealth, from which a person intends to consume $1 each year until death. For example, if one starts with an endowment of $20 and intends to consume $1 (after inflation) a year, the probability of sustainability is equal to the probability that the SPV is less than $20. This probability corresponds to the area under the curve to the left of the line at $20 on the *x*-axis. The probability of ruin is the area under the curve to the right of the $20 line. (Recall that the area under any curve on the graph totals 100 percent.) For any given case, a move to the left on the curve is equivalent to a decision to consume a higher proportion of wealth each year—because the individual is starting with a smaller nest egg and consuming one dollar—so, the probability that the plan is sustainable declines.

Figure 2. SPV of Retirement Consumption



The precise shape and parameters governing the SPV depend on the investment and mortality dynamics, but the general picture is remarkably consistent and similar to Figure 2. The SPV is defined over positive numbers, is right skewed, and is equal to zero at zero.

The two distinct curves in Figure 2 denote different cases. The solid-line curve that has more of its area to the left of the $20 line represents a scenario with a lower risk of ruin or shortfall. The dotted-line curve is a higher-risk case. For example, the solid curve could be a woman age 65 and the dotted curve could be a woman age 50. For the same portfolio (size and asset allocation), the woman age 65 has a lower probability of shortfall for any given consumption level because she has a shorter remaining life span for that consumption. Or the two curves could represent different people at the same age consuming the same amount but with different asset allocations; the allocation for the solid curve provides the lower risk of failing to earn enough to sustain the desired consumption level. What we want to know is the actual shape of Figure 2.

The analytic contribution of this article is implementation of a closed-form expression for the SPV defined by Equation 2b under the assumption that the total investment return, R_t, is generated by a lognormal distribution—that is, an exponential Brownian motion. This classical assumption has many supporters—from Merton (1975) to Rubinstein (1991). But even from an empirical perspective, Levy and Duchin (2004) found that the lognormal assumption "won" many of the "horse races" when plausible distributions for historical returns were compared. Furthermore, many popular optimizers, many asset allocation models, and much oft-quoted common advice are based on the classical Markowitz–Sharpe assumptions of lognormal returns. Therefore, for the remainder of this article, we follow this tradition.[4]

Analytic Formula for Sustainable Spending

Three important probability distributions allow us to derive a closed-form solution for sustainability. The first is the ubiquitous lognormal distribution, the second is the exponential lifetime distribution, and the last one is the—perhaps lesser-known—reciprocal gamma distribution. These three distributions merge together in the SPV. They allow us to solve the problem when investment return and date of death are risky or stochastic variables.

Lognormal Random Variable. The *investment total return*, R_t, between time t_0 and time t is said to be lognormally distributed with a mean of μ and standard deviation of σ if the expected total return is

$$E(R_t) = e^{\mu t}, \tag{3a}$$

the expected log return is

$$E[\ln(R_t)] = (\mu - 0.5\sigma^2)t, \tag{3b}$$

the log volatility is

$$E\{SD[\ln(R_t)]\} = \sigma\sqrt{t}, \tag{3c}$$

and the probability law can be written as

$$\text{prob}[\ln(R_t) < x] = N[(\mu - 0.5\sigma^2)t, \sigma\sqrt{t}, x], \tag{3d}$$

where $N(\cdot)$ denotes the cumulative normal distribution.

For example, a mutual fund or portfolio that is expected to earn an inflation-adjusted continuously compounded return of μ = 7 percent a year with a logarithmic volatility of σ = 20 percent has a $N(0.05,0.20,0)$ = 40.13 percent chance of earning a negative return in any given year. But if the expected return is a more optimistic 10 percent a year, the chances of losing money are reduced to $N(0.08,0.20,0)$ = 34.46 percent. Note that the expected value of lognormal random variable R_t is $e^{\mu t}$ but the median value (that is, geometric mean) is a lower $e^{(\mu-0.5\sigma^2)t}$. By definition, the probability that a lognormal random variable is less than its median value is precisely 50 percent. The gap between expected value $e^{\mu t}$ and median value $e^{(\mu-0.5\sigma^2)t}$ is always greater than zero, proportional to the volatility, and increasing in time. We will return to the *mean versus median* distinction later.

Exponential Lifetime Random Variable. The *remaining lifetime* random variable denoted by the letter T is said to be exponentially distributed with mortality rate λ if the probability law for T can be written as

$$\text{prob}(T > s) = e^{-\lambda s}. \tag{4a}$$

The expected value of the exponential lifetime random variable is equal to and denoted by

$$E(T) = \frac{1}{\lambda}, \tag{4b}$$

whereas the median value—which is the 50 percent mark—can be computed from

$$\text{Median}(T) = \frac{\ln(2)}{\lambda}. \tag{4c}$$

Note that the expected value is greater than the median value. For example, when λ = 0.05, the probability of living for at least 25 more years is $e^{-(0.05)(25)}$ = 28.65 percent and the probability of living for 40 more years is $e^{-(0.05)(40)}$ = 13.53 percent. The expected *lifetime* is 1/0.05 = 20 years, and the median lifetime is ln(2)/0.05 = 13.86 years.

Although human aging does not conform to an exponential or constant force of mortality assumption—which means that death would occur at a constant rate—for the purposes of estimating a sustainable spending rate, it does a remarkably good job when properly calibrated.

Reciprocal Gamma Random Variable. A random variable denoted by X is the *reciprocal gamma* (RG) variable distributed with parameters α and β if the probability law for X can be written as

$$\text{prob}(X \in dx) = \frac{x^{-(\alpha+1)}e^{(-1/x\beta)}}{\Gamma(\alpha)\beta^{\alpha}}dx. \tag{5}$$

Equation 5, which is the probability density function of the RG distribution, has two degrees of freedom, or free parameters, and is defined over positive numbers. The two defining (α,β) parameters must be greater than 0 for the PDF to properly integrate to an area under the curve value of 1. In fact, the parameter α must be greater than 1 for the expectation to be defined and must be greater than 2 for the standard deviation to be defined.

The denominator of Equation 5 includes a gamma function, Γ(α), that is defined and can be computed recursively as

$$\Gamma(\alpha) = \Gamma(\alpha - 1)(\alpha - 1). \tag{6}$$

The expected (mean) value—that is, first moment—of the RG distribution is

$$E(X) = [\beta(\alpha - 1)]^{-1}, \tag{7a}$$

and the second moment is

$$E(X^2) = [\beta^2(\alpha - 1)(\alpha - 2)]^{-1}. \tag{7b}$$

For example, within the context of this article, a typical parameters pair would be α = 5 and β = 0.03. In this case, the expected value of the RG variable would be 1/[(0.03)(4)] = 10.

The structure of the RG random variable is such that the probability an RG random variable is *greater* than some number x is equivalent to the probability that a gamma random variable is *less* than $1/x$. This fact is important (and quite helpful) because the gamma random variable is available in all statistical packages.

Main Result: Exponential Reciprocal Gamma

Our primary claim is that if one is willing to assume lognormal returns in a continuous-time setting, the stochastic present value in Figure 2 is the reciprocal gamma distributed in the limit. In other words, the probability that the SPV is greater than the initial wealth or nest egg, denoted by w, is

$$\text{prob}[SPV > w] = \text{GammaDist}\left(\frac{2\mu + 4\lambda}{\sigma^2 + \lambda} - 1, \frac{\sigma^2 + \lambda}{2} \mid \frac{1}{w}\right), \quad (8)$$

where GammaDist$(\alpha,\beta|\cdot)$ denotes the cumulative distribution function of the gamma distribution (in Microsoft Excel notation) evaluated at the parameter pair (α,β). The familiar μ and σ are the return and volatility parameters from the investment portfolio, and λ is the mortality rate. The expected value of the SPV is $(\mu - \sigma^2 + \lambda)^{-1}$.

For example, start with an investment (endowment, nest egg) of $20 that is expected to earn a 7 percent real return in any given year with a volatility (standard deviation) of 20 percent a year.[5] A 50-year-old (of any gender) with a median future life span of 28.1 years intends to consume $1 after inflation a year for the rest of his or her life. If the median life span is 28.1 years, then by definition, the probability of survival for 28.1 years is exactly 50 percent; so, the "implied mortality rate" parameter is $\lambda = \ln(2)/28.1 = 0.0247$. According to Equation 8, the probability of retirement ruin, which is the probability that the stochastic present value of $1 consumption is greater than $20, is 26.8 percent. In the language of Figure 2, if we evaluate the SPV at $w = 20$, the area to the right has a mass of 0.268 units. The area to the left—the probability of sustainability—has a mass of 0.732 units.[6]

In the random life span of $\lambda > 0$, our result is approximate, albeit correct to within two moments of the true SPV density. We will show that this issue is not significant. In the infinite horizon case of $\lambda = 0$, our result is *not an approximation*. It is a theorem that the SPV defined by Equation 2b is, in fact, the reciprocal gamma distributed.[7]

Numerical Examples

Our base case is a newly retired 65-year-old who has a nest egg of $1,000,000 which must last for the remainder of this individual's life. In addition to pensions, the retiree wants $60,000 a year in real dollars from this nest egg (which is $6 per $100 in the terms commonly used in practice). The $60,000 is to be created via a systematic withdrawal plan that sells off the required number of shares/units each month in a reverse dollar-cost-average strategy. These numbers are prior to any income taxes, and our results are for pretax consumption needs; in addition, we are not distinguishing between tax-sheltered and taxable plans, which is a different important issue.

The retiree wants to know whether the stochastic present value of the desired $60,000 income a year is *probabilistically less* than the initial nest egg of $1,000,000. If it is, the retiree's standard of living is sustainable. If the SPV of the consumption plan is larger than $1,000,000, however, the retirement plan is unsustainable and the individual will be "ruined" at some point, unless of course, he or she reduces consumption.

Table 2 provides an extensive combination of consumption/withdrawal rates for various ages based on our model in Equation 8 and based on exact mortality rates instead of the exponential approximation. The rates assume an all-equity portfolio with expected return of 7 percent and volatility of 20 percent. The time variable is determined by the first columns in Table 2—retirement age, the median age at death (based on actuarial mortality tables), and the implied hazard rate, λ, from this median value. The entries show the risk of ruin for annual spending rates ranging from $2 to $10 per $100 initial nest egg.

The first rows of Table 2 are for an endowment or foundation with an infinite horizon. The "exact" probability of ruin is derived directly from Equation 2b and ranges from a low of 15 percent ($2 spending) to a high of 92 percent ($10 spending).[8] According to Table 2, if the example person retiring at age 65 invests the $1,000,000 nest egg in this all-equity portfolio and withdraws the desired $60,000 a year, the exact probability of ruin is 25.3 percent.

The approximate answer from an estimated reciprocal gamma (ERG) formula based on an exponential future lifetime is, at 26.2 percent probability of ruin, slightly higher than the exact outcome. The gap between these two percentages is only 0.9 percentage points, however, which boosts our confidence in the model in Equation 8. Indeed, the differences throughout Table 2 between the results from using the model's assumption of exponential mortality—calibrated to the true median life span—and those from using the exact mortality table are reasonably small and do not materially change the assessment of the retiree's position. **Figure 3** illustrates for the base case with a range of consumption rates the approximation error from using the ERG formula when the "true" future lifetime random variable is more complicated.

Table 2. Reciprocal Gamma Approximation for Ruin Probability vs. Exact Results Using Correct Mortality Table

Retirement Age	Median Age at Death	Hazard Rate, λ		Real Annual Spending per $100 of Nest Egg $2.0	$3.0	$4.0	$5.0	$6.0	$7.0	$8.0	$9.0	$10.0
NA	Infinity	0.00%	Approx.:	15.1%	30.0%	45.1%	58.4%	69.4%	77.9%	84.4%	89.1%	92.5%
			Exact.:	15.1	30.0	45.1	58.4	69.4	77.9	84.4	89.1	92.5
			Diff.:	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	78.1	2.47	Approx.:	4.27	10.27	18.0	26.8	35.8	44.6	52.8	60.3	66.9
			Exact.:	3.04	9.10	17.8	27.7	37.8	47.2	55.5	62.6	68.5
			Diff.:	1.2	1.2	0.3	–0.9	–2.0	–2.6	–2.7	–2.3	–1.6
55	83.0	2.48	Approx.:	4.26	10.23	18.0	26.7	35.7	44.5	52.7	60.2	66.8
			Exact.:	2.83	8.95	18.0	28.7	39.6	49.9	59.0	66.7	73.0
			Diff.:	1.4	1.3	0.0	–2.0	–3.9	–5.4	–6.3	–6.5	–6.3
60	83.4	2.96	Approx.:	3.48	8.54	15.3	23.1	31.4	39.7	47.6	55.0	61.7
			Exact.:	1.82	6.36	13.7	22.9	32.9	42.7	51.7	59.6	66.4
			Diff.:	1.7	2.2	1.6	0.2	–1.5	–3.0	–4.1	–4.6	–4.6
65	83.9	3.67	Approx.:	2.64	6.68	12.27	18.9	26.2	33.7	41.1	48.3	54.9
			Exact.:	1.02	4.03	9.43	16.8	25.3	34.1	42.7	50.5	57.4
			Diff.:	1.6	2.7	2.8	2.1	0.9	–0.4	–1.5	–2.2	–2.5
70	84.6	4.75	Approx.:	1.61	4.73	8.95	14.2	20.1	26.5	33.0	39.5	45.8
			Exact.:	0.48	2.20	5.71	11.0	17.6	24.9	32.4	39.6	46.4
			Diff.:	1.3	2.5	3.2	3.2	2.6	1.6	0.6	–0.1	–0.6
75	85.7	6.48	Approx.:	1.07	2.90	5.69	9.32	13.6	18.5	23.6	29.0	34.4
			Exact.:	0.18	0.98	2.89	6.10	10.5	15.8	21.7	27.7	33.7
			Diff.:	0.9	1.9	2.8	3.2	3.1	2.6	1.9	1.2	0.7
80	87.4	9.37	Approx.:	0.52	1.47	3.00	5.10	7.71	10.8	14.2	18.0	21.9
			Exact.:	0.05	0.34	1.16	2.76	5.20	8.43	12.3	16.6	21.1
			Diff.:	0.5	1.1	1.8	2.3	2.5	2.3	1.9	1.4	0.8

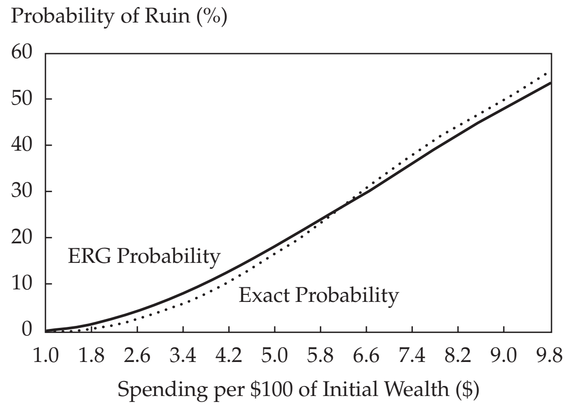
NA = not applicable.

Notes: Mean arithmetic portfolio return = 7 percent; standard deviation of return = 20 percent; mean geometric portfolio return = 5 percent. Differences may not be exact because of rounding.

For low consumption rates, the ERG formula slightly overestimates the probability of ruin and thus gives a more pessimistic picture. At higher consumption rates, the exact probability of ruin is higher than the approximation. Notice the relatively small gap between the two curves, which at its most is no more than 3–5 percent. The two curves are at their closest when the spending rate is between $5 and $7 per original $100.

Regardless of whether one uses the exact or the approximate methodology, a 25 percent chance of retirement ruin is unacceptable to most retirees. Table 2 indicates, however, that lowering the desired consumption or spending plan by $10,000 to a $50,000 systematic withdrawal plan can reduce the probability of ruin to 16.8 percent (in the exact method) or 18.9 percent (in the approximation). And if the spending plan is further reduced to $40,000, the probability of ruin shrinks to 9.4 percent (exact) and 12.3 percent (approximate). If the same individual were to withdraw $90,000, the probability of ruin would be 50.5 percent (exact) or 48.3 percent (approximate). The retiree or the financial planner can determine whether these odds are acceptable vis-à-vis the retiree's tolerance for risk.

Figure 3. Approximation vs. Exact Probability That Given Spending Rate Is Not Sustainable



Note: Age = 65; mean arithmetic portfolio return = 7 percent; standard deviation of return = 20 percent.

To understand the intuition behind the numbers, recall that the mean or expected value of the SPV of $1 of real spending is $1/(\mu - \sigma^2 + \lambda)$, where μ and σ are the investment parameters and λ is the mortality rate parameter induced by a given median remaining lifetime. For a 65-year-old of either sex, the median remaining lifetime is 18.9 years (83.9 median age of death in Table 2 minus actual age of 65) according to the RP-2000 Society of Actuaries mortality table. To obtain the 50 percent probability point with an exponential distribution, we solve for $e^{-18.9\lambda} = 0.5$, which leads to $\lambda = \ln(2)/18.9 = 0.0367$ as the implied rate of mortality. The mean value of the SPV for μ = 7 percent and σ = 20 percent works out to $1/(0.07 - 0.04 + 0.0367)$, which is an average of $15 for the SPV per dollar of desired consumption. Thus, if the retiree intends to spend $90,000 a year, it should come as no surprise that a nest egg of only 11 times this amount is barely sustainable on average. Note that the expected value of the SPV decreases in μ and λ and increases in σ. Higher mean is good, higher volatility is bad, and the benefit of a higher mortality rate comes from reducing the length of time over which the withdrawals are taken.

Effects of Investment Strategies

We are not entering the debate about what are the "right" values for return expectations because our work makes no contribution to answering that important but contentious question. But we can use our model to show the effect of various portfolio composition and return assumptions. The portfolio in Table 2 is an all-equity portfolio with mean return of 7 percent and volatility of 20 percent. As expected, if the mean return is higher or the volatility lower, with all else held constant, the sustainability improves, and vice versa. What happens, however, if we change both parameters in the same direction, which is what we normally expect in an efficient financial market?

Consider a common portfolio that is 50 percent equity and 50 percent bonds, which we will say has a mean arithmetic return of 5 percent and volatility of 12 percent. **Table 3** shows the probabilities of unsustainable spending for various ages and spending rates per $100 of nest egg for such a balanced portfolio.

Consider first the base case of a newly retired 65-year-old who has a nest egg of $1,000,000, from which the retiree wants $60,000 a year. In this case, Table 3 shows that the risk of ruin is a bit lower, at 24 percent, than it was for the all-equity portfolio but not much lower. Most retirees would be unhappy with that chance. Again, cutting consumption lowers the probability of ruin; a reduction to $40,000 would lower the chance of ruin to a more acceptable 9 percent. If the person waits until age 70 to retire and wants to withdraw $60,000 a year as planned, the risk of ruin drops to 17.6 percent.

Table 3. Ruin Probability Approximation for Balanced Portfolio of 50 Percent Equity and 50 Percent Bonds

			Real Annual Spending per $100 of Initial Nest Egg								
Retirement Age	Median Age at Death	Hazard Rate, λ	$2.00	$3.00	$4.00	$5.00	$6.00	$7.00	$8.00	$9.00	$10.00
Endowment	Infinity	0.00%	6.7%	24.9%	49.0%	70.0%	84.3%	92.5%	96.6%	98.6%	99.4%
50	78.1	2.47	1.8	6.4	14.0	24.0	35.2	46.3	56.8	66.0	73.8
55	83.0	2.48	1.8	6.3	14.0	24.0	35.1	46.2	56.7	65.9	73.7
60	83.4	2.96	1.5	5.2	11.6	20.1	29.9	40.1	50.0	59.1	67.2
65	83.9	3.67	1.1	4.0	9.0	15.8	24.0	32.8	41.8	50.5	58.5
70	84.6	4.75	0.8	2.8	6.3	11.4	17.6	24.7	32.2	39.8	47.2
75	85.7	6.48	0.5	1.7	3.9	7.2	11.4	16.3	21.9	27.8	33.9
80	87.4	9.37	0.3	0.9	2.0	3.8	6.2	9.1	12.5	16.3	20.5

Note: Mean arithmetic portfolio return = 5 percent; standard deviation of return = 12 percent; mean geometric portfolio return = 4.28 percent.

In general, for cases that are toward the lower left of Tables 2 and 3, the probability of ruin falls somewhat with a lower-risk/lower-return portfolio. In contrast, for cases that are toward the upper right of the tables, the risk of ruin is higher with a lower-risk/lower-return portfolio. The reason for these opposing effects lies in the nature of the consumption pattern. If a retiree wants a lot of income from a portfolio, relative to the size of the portfolio and/or relative to his or her expected remaining lifetime, then the risk of shortfall can be reduced only by gambling on a high-risk/high-return portfolio. Changing to a high-risk/high-return portfolio does not give the retiree a satisfactory reduction in the risk of ruin, however, because the values in the upper right of Table 2 are all unacceptably high from any point of view. If the retiree will settle for a more reasonable level of consumption—say, no more than $6 per $100 at age 65—the more balanced portfolio also reduces the risk of ruin. (Note that for Table 3, we reduced volatility by 8 percent but return by only 2 percent. Changes in return have more effect than changes in volatility.)

This message is particularly unsettling for endowments that are required to pay out in perpetuity. Their payout rates are almost always in the range of 4–6 percent of principal. But the odds of maintaining the real value of that payout are poor for either a balanced or an all-equity portfolio, with the probability of ruin ranging from 45 percent to 84 percent, depending on the payout and the asset allocation. An endowment can always maintain a payout of some percentage of the market value of assets in perpetuity, but our results are saying that the *real value* of whatever is paid out will probably have to be reduced, thus providing less and less real value for student scholarships, research projects, or whatever is funded by the endowment. Lest the reader think this judgment is simply a case of unreasonably pessimistic investment return assumptions, consider the following. Even with a long-run real return expected to be 9 percent with a standard deviation of 16 percent—a remarkable performance if anyone could maintain it over the long run—the risk of ruin for a perpetual endowment is 9.5 percent for a perpetual payout of real $4 per $100 of principal today. The risk of ruin rises to 19.6 percent if the payout is $5 and to 32.4 percent if the payout is $6.

In **Table 4**, we pose the question of which action helps the base-case 65-year-old more—reducing consumption or changing the investment portfolio. The "Portfolio" column headings consist of possible combinations of mean and volatility to represent various investment strategies; they range from low risk and low return to high risk and high return. Down the side is consumption per $100 of nest egg. At every level of consumption, Table 4 shows that the choice of investment portfolio does not matter much. At levels of $2 or $3 per $100, any portfolio gives a low probability of ruin. At $4 per $100, the individual's tolerance for risk could begin to affect the portfolio choice. Once the consumption rises to $5 per $100, the probabilities of ruin would be unacceptable to most retirees. No matter what reasonable portfolio is chosen, asset allocation will not turn a bad situation into a good one.

Another interesting insight comes from examining the interplay between the three main parameters in our formula. Increasing the fixed mortality rate, λ, by 100 bps—which reduces the median future lifetime from $\ln(2)/\lambda$ to $\ln(2)/(\lambda + 0.01)$—obviously reduces the probability of retirement ruin, all else being equal. The same reduction can also be achieved by increasing the portfolio return by 200 bps together with increasing the portfolio variance by 100 bps. Recall that the (α,β) parameter arguments in Equation 8 can be expressed as a function of $(\mu + 2\lambda)$ and $(\sigma^2 + \lambda)$. Thus, having a longer life span is interchangeable with decreasing the portfolio return or increasing portfolio variance.

Table 4. Probability of Ruin for Various Portfolios at Age 65

Consumption per $100 of Nest Egg	Portfolio: Mean Arithmetic Return and [Volatility]				
	4% [10%]	5% [12%]	6% [15%]	7% [17%]	8% [20%]
$2.00	1.5%	1.1%	1.3%	1.2%	1.6%
$3.00	5.0	4.0	4.1	3.8	4.5
$4.00	11.1	9.0	8.8	8.0	8.8
$5.00	19.1	15.8	15.1	13.7	14.4
$6.00	28.4	24.0	22.5	20.4	20.8
$7.00	38.2	32.8	30.6	27.7	27.6
$8.00	47.8	41.8	38.8	35.3	34.7

Another perspective on the issue can be gained by fixing a "ruin tolerance" level and then inverting Equation 8 to solve for the level of spending that satisfies the given probability. Leibowitz and Henriksson (1989) advanced this idea within the context of a static portfolio asset allocation. Browne (1995, 1999) and then Young (2004) solved the dynamic versions of this portfolio control problem for a finite and a random time horizon, respectively. The inversion process is relatively easy because a number of software packages have a built-in function for the inverse of the gamma function in which the argument is the probability rather than the spending rate.

Table 5 takes this inverted approach by solving for the sustainable spending rate that results in a given probability of ruin for the same set of possible portfolios used in Table 4. On the one hand, if a 65-year-old retiree is willing to assume or "live with" a ruin probability of only 5 percent (Panel B), which means that he desires a 95 percent chance of sustainability, the most he can consume from a balanced portfolio with mean return of 5 percent and volatility of 12 percent is $3.24 per initial nest egg of $100. On the other hand, if he is willing to tolerate a 10 percent chance of ruin (Panel A), the maximum consumption level increases from $3.24 to $4.17 per $100. The higher the ruin-tolerance level, the more he can consume.[9] An increase in return or a decrease in volatility always raises sustainable consumption, but if risk and return tend to move together in the long run, as is generally observed in practice, changes in asset allocation will not have a major effect.

Conclusion and Next Steps

Our analysis using the stochastic present value provides an analytic method for assessing the sustainability of retirement plans and offers new insights into, in particular, retirement longevity risk.

The distinction between Monte Carlo simulations and the analytical techniques promoted in this article is more than simply a question of academic tastes.[10] Although simulations will continue to have a legitimate and important role in the field of wealth management, our simple formula can serve as a test and calibration tool for more complex simulation. It can also explain the link between the three fundamental variables affecting retirement planning: spending rates, uncertain longevity, and uncertain returns. The formula makes clear that increasing the mortality hazard rate—which is equivalent to aging—while holding the probability of ruin constant has the same effect as increasing

Table 5. Sustainable Spending Rate That Results in a Given Probability of Ruin for Various Portfolios at Different Retirement Ages

		Portfolio: Arithmetic Return, [Volatility], and (Geometric Return)					
Current Age	Hazard Rate	3.00% [10.00%] (2.50%)	4.00% [10.00%] (3.50%)	5.00% [12.00%] (4.28%	6.00% [15.00%] (4.88%)	7.00% [17.00%] (5.56%)	8.00% [20.00%] (6.00%)
A. Probability of ruin 10%							
Endowment	0.00%	$1.22	$1.95	$2.24	$2.22	$2.37	$2.20
50	2.47	2.54	3.20	3.52	3.55	3.72	3.56
55	2.48	2.55	3.21	3.52	3.55	3.72	3.57
60	2.96	2.81	3.47	3.79	3.82	3.99	3.84
65	3.67	3.20	3.85	4.17	4.20	4.38	4.23
70	4.75	3.79	4.44	4.75	4.80	4.97	4.82
75	6.48	4.74	5.38	5.70	5.75	5.92	5.77
80	9.37	6.33	6.96	7.28	7.33	7.51	7.37
B. Probability of ruin 5%							
Endowment	0.00%	$0.99	$1.64	$1.86	$1.76	$1.84	$1.64
50	2.47	1.95	2.52	2.77	2.73	2.84	2.64
55	2.48	1.96	2.53	2.77	2.74	2.84	2.65
60	2.96	2.15	2.72	2.96	2.93	3.04	2.85
65	3.67	2.44	3.00	3.24	3.22	3.32	3.13
70	4.75	2.88	3.43	3.67	3.66	3.76	3.58
75	6.48	3.58	4.12	4.37	4.36	4.47	4.28
80	9.37%	4.76	5.29	5.54	5.53	5.65	5.46

the portfolio rate of return and decreasing the portfolio volatility. An implication is that females, who are expected to live three to five years longer than males, on average, and thus have a lower rate of death at any given age, should be spending less in order to maintain the same (low) probability of ruin that males need.

Even with the most tolerant attitude toward the risk of ruin, a retiree should be spending no more than, with the notation used in this article, $(\mu - \sigma^2 + \lambda)$ percent of the initial nest egg, where λ is ln(2) divided by the median future lifetime. This spending rate should be sustainable, on average, because the expected value of $1 consumption for life is $(\mu - \sigma^2 + \lambda)^{-1}$.

Here is another way to think about average sustainability. Note that $\mu - \sigma^2$ is even lower than the (continuously compounded) geometric mean $\mu - 0.5\sigma^2$. If the arithmetic mean return is 7 percent, then the geometric mean return is $\mu - 0.5\sigma^2 = 5$ percent and the quantity $\mu - \sigma^2 = 3$ percent, for $\sigma = 20$ percent volatility. Thus, a retiree who is satisfied with average sustainability can plan to spend only 3 percent plus an additional 0.693 divided by her or his median remaining lifetime. A median of 10 more retirement years can add 6.9 percentage points to spending; a median of 20 and 30 more retirement years adds, respectively, 3.5 percentage points and 2.3 percentage points. For an endowment or foundation, $\lambda = 0$ and, therefore, average sustainability can be achieved only by spending no more than 3 percent of contributed capital. Of course, if the assumed 20 percent volatility can be reduced by further diversification, these static spending rates can obviously be increased. But then again, using a higher volatility might be a prudent hedge against model misspecification—specifically, the "jump and crash" risk that is not adequately captured in a lognormal distribution.

We want to stress that we are not advocating ruin minimization as a normative investment strategy. Notwithstanding its inconsistency with rational utility maximization, Browne (1999) and Young have both documented the uncomfortably high degrees of leverage such a dynamic policy might entail. Rather, we believe that the probability of retirement ruin is a useful risk metric that can help retirees understand the link between their desired spending patterns, retirement age, and the current composition of their investment portfolios.

Indeed, the concept of a stochastic present value of a retirement plan can be used beyond the limited scope of computing probabilities of ruin. For example, one can use this idea to investigate the impact of including payout annuities or nonlinear instruments in a retiree's (or endowment's) portfolio. Similarly, the SPV can be used to compare the relative tax efficiency of various asset location decisions for retirement income products and the role of life annuities in increasing the sustainability of a given spending rate. For example, we have found that including zero-cost collars in the retiree's portfolio (i.e., selling out-of-the-money calls whose funds are then used to purchase out-of-the-money puts) shifts the SPV toward zero, which reduces the probability of ruin and increases the sustainability of the portfolio. In summary, we urge the financial industry to focus on designing products that maximize income sustainability over a random retirement horizon.

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Notes

1. The National Association of College and University Business Officers endowment survey conducted in 2004 showed that the median endowment spending rate in 2003 was 5.0 percent of assets, with the 10th percentile being 4.0 percent and the 90th percentile being 6.4 percent.
2. Using several free Web-based simulators, we ran some case studies and found wide variations in the suggested "nest egg" needed to support a comfortable retirement. A similar concern about the variation in simulation outcomes—which was misinterpreted as a criticism of the Monte Carlo method—was echoed recently by McCarthy (2002/2003).
3. The spreadsheet is available by selecting this article from the November/December contents page on the *FAJ* website at www.cfapubs.org.
4. We provide an analysis of the effect of this assumption in an online technical appendix.
5. We discuss the question of reasonable return distribution assumptions later.
6. The more technically inclined readers might want more than simply a formula. A proof that Equation 8 is the proper distribution of the stochastic present value is based on moment-matching techniques and the partial differential equations for the probability of ruin based on Equation 2b. We believe that a variant of this result can be traced back to Merton. For more details, proofs, and restrictions, see Milevsky (1997), Browne (1999), or Milevsky (forthcoming 2006)—specifically, the actuarial, financial, and insurance references contained in this last article.

7. For those readers who remain unconvinced that what is effectively the "sum of lognormals" can converge to the inverse of a gamma distribution, we suggest they simulate the SPV for a reasonably long horizon and conduct a Kolmogorov–Smirnov goodness-of-fit test of the inverse of these numbers against the gamma distribution with the parameters given by $\alpha = (2\mu + 4\lambda)/(\sigma^2 + \lambda) - 1$ and $\beta = (\sigma^2 + \lambda)/2$. As long as the volatility parameter, σ, is not abnormally high relative to the expected return, μ, they will get convergence of the relevant integrand.
8. By "exact" probability of ruin, we mean the outcome from using a complete actuarial mortality table starting at age 65 to discount all future cash flows rather than using the exponential lifetime approximation. See Huang, Milevsky, and Wang (2004) for more details about the accuracy of such an approximation based on partial differential equation (PDE) methods.
9. A variant of this "probabilistic spending" rule was designed by one of the authors and was recently implemented by the Florida State Board of Administration for its billion-dollar Lawton Chiles Endowment Fund. Each year, the trustees of the fund compute the probability of preserving its real value and then adjust spending up or down accordingly. See www.sbafla.com/pdf/funds/LCEF_TFIP_2003_02_25.pdf for more information.
10. For example, Whitehouse (2004) described the benefits of analytic PDE-based solutions over Monte Carlo simulations.

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